Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021–8928(99)00005-2
J. Appl. Maths Mechs, Vol. 63, No. 1, pp. 25–30, 1999

0021–8928/99/$—see front matter

THE EQUATIONS OF MOTION OF A RIGID BODY WITHOUT PARAMETRIZATION OF ROTATIONS†

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(*Received* 5 *November* 1997)

New dynamic equations are proposed for a rigid body, without using local parametrization of the rotation group to describe the rotational part of the motion. A simple system of differential-algebraic equations, well suited for constructing the equations of motion of articulated bodies, is obtained. © 1999 Elsevier Science Ltd. All rights reserved.

There are several ways to parametrize the rotational motions of a rigid body, most frequently making use of Euler or Bryant angles. The problem with such parametrizations is that they do not provide a global map for the rotation group SO(3). Various devices have been proposed to remedy this situation. An example is the use of Rodrigues–Hamilton parameters [1, 2] or, equivalently, quaternions [3] or dual numbers [4]. Extensive bibliographies for the parametrization of rotations can be found in [4, 5].

In the 1960s, Arnol'd [6] proposed to base the notation of the dynamic equations exclusively on the SO(3) group structure. Chevallier [7, 8] generalized these ideas, making exclusive use of the geometrical and differential structure of the Lie group of displacements D. Utilizing the associated representation of the Lie group D in its Lie algebra, he obtained a simple synthetic formulation of the dynamic equations.

In this paper we propose to avoid the parametrization of rotations by treating an arbitrary rotation directly as an element of an ensemble of non-singular 3×3 matrices, which is an open set in the normed vector space of 3×3 matrices.

The condition $RR^T = R^TR = I$ (where R denotes the matrix of a rotation and I is the identity matrix) is considered as a constraint, which is taken into account by introducing six Lagrange multipliers. These multipliers can be represented by a symmetric 3×3 matrix.

In this formulation of the problem, we begin with a computation of the kinetic energy, and then write down the Lagrange equations.

Finally, the mechanical meaning of the Lagrange multipliers will be explained and an illustrative example presented.

1. THE EQUATIONS OF MOTION

Let S be a rigid body moving in a fixed Galilean reference system with frame R_0. Let O be the origin of this frame, and let A and M be two points of S whose positions at the initial time are A_0 and M_0, respectively. At each instant of time we have the equality

$$\mathbf{r}(t) = \mathbf{a}(t) + \mathbf{b}(t) = \mathbf{a}(t) + R(t)\mathbf{b}_0$$

where $\mathbf{r}(t)$ is the radius vector of the point M, $\mathbf{a}(t)$ is the translational displacement vector (the radius vector of the point A), $R(t)$ is the rotation matrix and $\mathbf{b}$ is a vector with origin at A and end at M.

The velocity of the point M in R_0 may be written as

$$\mathbf{V}(M/R_0) = \dot{\mathbf{a}}(t) + \dot{R}(t)\mathbf{b}_0 = \dot{\mathbf{a}}(t) + \dot{R}(t)[R(t)]^{-1}\mathbf{b}(t) \tag{1.1}$$

where a dot over a letter denotes differentiation with respect to time in R_0. The aim of this paper is to express the equation of motion of the rigid body S in terms of R and $\mathbf{a}$, given equations of constraints, allowing for the fact that R is a rotation matrix: $RR^T = I$, by introducing Lagrange multipliers.

The expression for the kinetic energy. Let m be the mass of S. Consider the tensor

$$K = \int_S \mathbf{b} \otimes \mathbf{b} \ dm$$

†*Prikl. Mat. Mekh.* Vol. 63, No. 1, pp. 30–36, 1999.

where $\otimes$ denotes tensor multiplication. The tensor K is related to the inertia tensor of the body at the point A, which will be denoted by J. To be precise: if K_0 denotes the value of K at the initial time, that is

$$K_0 = \int_S b_0 \otimes b_0 dm$$

we have the following lemma.

Lemma 1. At each instant of time, the tensors K and K_0 satisfy the equality $K(t) = R(t)K_0R^T(t)$; in addition, J and K satisfy the relation

$$K = \frac{1}{2}(trJ)I - J$$

Proof. Making the change of variables $\mathbf{b} = R\mathbf{b}_0$, whose Jacobian is equal to unity, we obtain

$$K = \int_{S_0} R\mathbf{b}_0 \otimes R\mathbf{b}_0 dm$$

Since $R\mathbf{b}_0 \otimes R\mathbf{b}_0 = R(\mathbf{b}_0 \otimes \mathbf{b}_0)R^T$, the first part of the lemma is proved. The classical definition of the inertia tensor of the body at the point A is

$$J = \mathrm{tr}\left(\int_S \mathbf{b}\otimes\mathbf{b}\ dm\right)I - \int_S \mathbf{b}\otimes\mathbf{b}\ dm$$

Consequently, $J = (\mathrm{tr}K)I - K$. Evaluating the traces of the right- and left-hand sides of this equality, we obtain $\mathrm{tr}J = 2\mathrm{tr}K$, whence follows the second part of the lemma.

Let G be the position of the centre of mass of the body S at the time t, and G_0 the same position at the initial time. We can then establish the following property.

Property 1. The kinetic energy of a body moving relative to the frame R_0 is

$$W\left(\frac{S}{R_0}\right) = \frac{1}{2}\left(\mathrm{tr}\left(\dot{R}K_0\dot{R}^T\right) + m\|\dot{\mathbf{a}}\|^2\right) + m\left\langle\dot{\mathbf{a}}, \dot{R}\mathbf{C}_0\right\rangle$$

where $\langle\ ,\ \rangle$ denotes the standard scalar product and the vector $\mathbf{c}_0$ has its origin at A_0 and its other end at G_0.

Proof. By the definition of the kinetic energy of S relative to the frame R_0, we have

$$2W = \int_S \left\|\mathbf{V}(M/R_0)\right\|^2 dm$$

Evaluating the integrand on the basis of (1.1), we obtain

$$2W = m\|\dot{\mathbf{a}}\|^2 + 2m\left\langle\dot{\mathbf{a}}, \dot{R}R^T\mathbf{c}\right\rangle + \mathrm{tr}\left(\dot{R}R^TKR\dot{R}^T\right)$$

Using Lemma 1, on the one hand, and the equality $\mathbf{c}_0 = R^T\mathbf{c}$, on the other, we obtain the required property.

The virtual work of the external forces applied to the body. In view of the expression for $\mathbf{r}$, the virtual displacement of the point M may be written as

$$\delta\mathbf{r} = \delta R\mathbf{b}_0 = \delta\mathbf{a}$$

The virtual displacement of the whole body may be characterized by the pair $(\delta R, \delta\mathbf{a})$.

Property 2. For any forces applied to S, a matrix Q and a vector $\mathbf{L}$ exist such that the virtual work of the forces to implement the virtual displacement $(\delta R, \delta\mathbf{a})$ may be expressed as follows:

$$T^\nu = \mathrm{tr}\left(Q\delta R^T\right) + \langle L, \delta\mathbf{a}\rangle$$

This result is obvious, since the virtual work T^{υ} is a linear form in the space of virtual displacements (δR, $\delta \mathbf{a}$). Consequently, a matrix Q and a vector $\mathbf{L}$ exist satisfying the desired requirements.

Example. Let us determine $\mathbf{L}$ and Q for the volume density of forces $\mathbf{f}$ acting on the body S. We have

$$T^{\upsilon} = \int_S \langle \mathbf{f}(\mathbf{r}), \delta\mathbf{r} \rangle dv = \int_S \langle \mathbf{f}(\mathbf{r}), \delta R R^T \mathbf{b} + \delta\mathbf{a} \rangle dv$$

Comparing this expression with that which figures in Property 2, we obtain

$$\mathbf{L} = \int_S \mathbf{f}(\mathbf{r}) dv, \quad Q = \left(\int_S \mathbf{f}(\mathbf{r}) \otimes \mathbf{b}\, dv \right) R$$

The vector $\mathbf{L}$ is the principal vector of the forces applied to S, and the matrix Q contains information on the moments of these forces relative to A.

Lagrange equations. We first note that $\partial W/\partial R = 0$ and $\partial W/\partial \mathbf{a} = 0$. On the other hand

$$\frac{d}{dt}\left(\frac{\partial W}{\partial \dot{R}}\right) = \ddot{R} K_0 + m\ddot{\mathbf{a}} \otimes \mathbf{c}_0, \quad \frac{d}{dt}\left(\frac{\partial W}{\partial \dot{\mathbf{a}}}\right) = m\left(\ddot{R}\mathbf{c}_0 + \ddot{\mathbf{a}}\right)$$

Taking into account the expression for the virtual work and the constraint equation $RR^T = I$, whose variation is

$$(\delta R) R^T + R(\delta R)^T = 0 \tag{1.2}$$

we can write the equations of motion in the form

$$\ddot{R} K_0 + m\ddot{\mathbf{a}} \otimes \mathbf{c}_0 = Q + R\Lambda, \quad m\left(\ddot{\mathbf{a}} + \ddot{R}\mathbf{c}_0\right) = \mathbf{L} \tag{1.3}$$

where Λ is a symmetric 3×3 matrix, whose six elements are Lagrange multipliers, in accordance with the constraint equations (1.2).

Interpretation of Λ. Let us consider the special case of a rigid body with a fixed point O. In addition, we assume $Q = 0$ for simplicity. Under these conditions the following lemma holds.

Lemma 2. For a rigid body with a fixed point O, if the moment of the external forces about that point is zero, then the matrix of Lagrange multipliers is negative.

Proof. By the assumptions of the lemma, we can rewrite the equations of motion as

$$\ddot{R} K_0 = R\Lambda, \quad R^T R = I$$

Differentiating this equality twice and using the equality $R^T \ddot{R} = \Lambda K_0^{-1}$, which follows from the equations of motion, we get

$$K_0^{-1}\Lambda + \Lambda K_0^{-1} = -2\dot{R}^T \dot{R}$$

Since the matrix Λ is symmetric, it is diagonalizable and its eigenvalues are real.
To prove that this matrix is negative, it will suffice to show that all its eigenvalues are negative.
Let λ be an eigenvalue of Λ and let $\mathbf{u}$ be a corresponding eigenvector. Then

$$\left\langle K_0^{-1}\Lambda u,\ u\right\rangle + \left\langle \Lambda K_0^{-1} u,\ u\right\rangle = -2\left\langle \dot{R}^T \dot{R} u,\ u\right\rangle$$

Taking the symmetry of Λ into account, we have

$$\lambda\left\langle K_0^{-1} u,\ u\right\rangle = -\left\langle \dot{R}^T \dot{R} u,\ u\right\rangle$$

On the other hand, K_0 is a positive matrix, and therefore the form $\langle K_0^{-1}u, u\rangle$ is strictly positive. In addition, the form $\langle \dot{R}u, \dot{R}u \rangle$ is also positive, whence it follows that λ is negative.

This result enables us to interpret Λ as the internal stresses due to the inextensibility conditions for a rigid body. To be precise, we have the following property.

Property 3. Let S be an absolutely rigid body rotating about a fixed point O relative to the frame R_0 by inertia. Then the matrix of Lagrange multipliers may be written as

$$\Lambda = -R^T\left(\int_S \sigma\, dv\right)R$$

where σ is the tensor of the Cauchy internal stresses.

Let us consider the problem from the standpoint of the theory of deformable media. When there are no external forces, an arbitrary virtual displacement gives rise to virtual work of the forces of inertia, equal and opposite to the virtual work of the internal stresses: $T_a^v = -T_i^v$. Since, on the one hand

$$T_i^v = \int_S \operatorname{tr}\left(\sigma\frac{\partial \delta\mathbf{r}}{\partial \mathbf{r}}\right)dv$$

it follows, in view of the expression for $\delta\mathbf{r}$, that

$$T_i^v = \int_S \operatorname{tr}\left(\sigma R\delta R^T\right)dv$$

(we have used the fact that σ is a symmetric tensor).

On the other hand, we have

$$T_a^v = \operatorname{tr}\left(P\delta R^T\right), \quad P = \frac{d}{dt}\left(\frac{\partial W}{\partial R}\right) - \frac{\partial W}{\partial R} = R\Lambda$$

Comparing the virtual work principle formulated above for an absolutely rigid body and for an elastic body, we arrive at the desired conclusion.

Remark. We have thus established that the form $R\Lambda R^T$ is the volume average of the internal stresses in the body S.

2. APPLICATION

With the help of a classical but non-trivial example, we will show how the equations of motion obtained above may be interpreted, without recourse to parametrization of rotation.

Consider a solid of revolution S with fixed mass centre O and vector $(0, \mathbf{z})$ directed along the axis of symmetry. To $\mathbf{z}$ we add two unit vectors $\mathbf{x}$ and $\mathbf{y}$ so as to form the frame of principal axes of inertia of S.

Let B be the moment of inertia relative to any axis passing through O and situated in the plane (O, x, y) and let C be the moment of inertia about the axis of symmetry. The inertia tensor is given by the expression

$$J = BI + (C - B)\mathbf{z} \otimes \mathbf{z}$$

Let $(O; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ be the initial position of the frame of principal axes of inertia of S. At each instant of time, the motion is defined by a matrix of rotation R which takes the frame $(O; \mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ into a frame $(O; \mathbf{x}, \mathbf{y}, \mathbf{z})$. The equations of motion are

$$\ddot{R}K_0 = R\Lambda, \quad R^T R = I \tag{2.1}$$

$$\left(K_0 = IC/2 + (B - C/2)\mathbf{z}_0 \otimes \mathbf{z}_0\right)$$

Differentiating the second equation in (2.1), we see that $R^T\dot{R}$ is an anti-symmetric matrix. Consequently, a vector exists, which we denote by $\boldsymbol{\omega}$, such that $\dot{R} = Rj(\boldsymbol{\omega})$, where $j(\boldsymbol{\omega})$ is the anti-symmetric mapping defined by

$$j(\boldsymbol{\omega})\mathbf{v} = \boldsymbol{\omega} \times \mathbf{v}$$

(the right-hand side is a vector product). The vector $R\boldsymbol{\omega}$ is the classical vector of instantaneous angular velocity. Differentiating the second equation of (2.1) once more and substituting into the first equation, we obtain

$$(j(\boldsymbol{\omega}))^2 K_0 + j(\dot{\boldsymbol{\omega}})K_0 = \Lambda$$

Since Λ is a symmetric matrix, it follows, subtracting the transpose of this relationship from the relationship itself, that

$$[j(\boldsymbol{\omega})]^2 K_0 - K_0[j(\boldsymbol{\omega})]^2 + j(\dot{\boldsymbol{\omega}})K_0 + K_0 j(\dot{\boldsymbol{\omega}}) = 0 \tag{2.2}$$

Thus, the Lagrange multipliers have been eliminated. Using the properties of the vector product and the trace operator, we can verify that

$$[j(\boldsymbol{\omega})]^2 K_0 - K_0[j(\boldsymbol{\omega})]^2 = j\big((K_0\boldsymbol{\omega})\times\boldsymbol{\omega}\big),$$

$$j(\dot{\boldsymbol{\omega}})K_0 + K_0 j(\dot{\boldsymbol{\omega}}) = j\big[\big(\mathrm{tr}(K_0)I - K_0\big)\dot{\boldsymbol{\omega}}\big].$$

By virtue of these arguments, using also Lemma 1, we can reduce Eq. (2.2) to the form

$$J_0\dot{\boldsymbol{\omega}} - (J_0\boldsymbol{\omega})\times\boldsymbol{\omega} = 0 \tag{2.3}$$

in which we easily recognize the equation for the angular momentum.

Evaluating the scalar product of Eq. (2.3) by $\mathbf{z}_0$, we obtain $\langle \mathbf{z}_0, \dot{\boldsymbol{\omega}}\rangle = 0$. Consequently, $\langle \mathbf{z}_0, \boldsymbol{\omega}\rangle = \mathrm{const}$ and

$$\dot{\boldsymbol{\omega}} = (C/B - 1)\langle \mathbf{z}_0, \boldsymbol{\omega}_0\rangle j(z_0)\boldsymbol{\omega} \tag{2.4}$$

where we have put $\boldsymbol{\omega}(0) = \boldsymbol{\omega}_0$.

Taking this initial condition into consideration, we obtain the solution of Eq. (2.4)

$$\boldsymbol{\omega} = E(t)\boldsymbol{\omega}_0, \quad E(t) = \exp\big[t(C/B - 1)\langle \mathbf{z}_0, \boldsymbol{\omega}_0\rangle j(\mathbf{z}_0)\big]$$

where $E(t)$ is a rotation matrix.

It remains to integrate the equation

$$dR/dt = Rj(E\boldsymbol{\omega}_0)$$

A more general form of this equation has already been investigated in [9–11], but the result is too general and implicit in nature. In the special case considered here, a simple explicit solution can be presented. Indeed, since E is a rotation matrix, it follows that

$$j(E\boldsymbol{\omega}_0) = Ej(\boldsymbol{\omega}_0)E^{-1}$$

Using the expression for the derivative dE/dt, we obtain

$$d(RE)/dt = RE\big[j(\boldsymbol{\omega}_0) + (C/B - 1)\langle \mathbf{z}_0, \boldsymbol{\omega}_0\rangle j(\mathbf{z}_0)\big]$$

Taking the initial condition into consideration, we obtain a solution of this equation, which finally yields

$$R(t) = \exp\left[t\left(j(\boldsymbol{\omega}_0) + \left(\frac{C}{B} - 1\right)\langle \mathbf{z}_0, \boldsymbol{\omega}_0\rangle j(\mathbf{z}_0)\right)\right]\times$$
$$\times\exp\left[-t\left(\frac{C}{B} - 1\right)\langle \mathbf{z}_0, \boldsymbol{\omega}_0\rangle j(\boldsymbol{\omega}_0)\right]$$

Consequently, the motion of the rigid body is the product of two rotations: 1) about the initial position $(O;\ \mathbf{z}_0)$ of the axis of symmetry of the rigid body; 2) about an axis passing through the centre of mass and directed along the vector

$$\boldsymbol{\omega}_0 + (C/B - 1)\langle \mathbf{z}_0, \boldsymbol{\omega}_0\rangle \mathbf{z}_0.$$

3. CONCLUSION

Considering a rotation as an element of the family of 3×3 non-singular matrices satisfying the constraint $R^TR = RR^T = I$, we have been able to derive a simple structure for the dynamic equations of a rigid body. Allowance for the fact that a rigid body is not deformable is made through a symmetric matrix Λ of Lagrange multipliers.

We have shown that the matrix Λ is related to the volume average of the Cauchy internal stresses in the body. The simplicity of the equations obtained yields a visually intuitive illustration of the solution of the Euler–Poinsot problem. The equations may be extended to a three-body system The formalism constructed is eminently suited for formulating a procedure for constructing the appropriate equations. In this connection, we note that this formalism has recently been developed to simulate the virtual reality of a system of several linked bodies [12].

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Translated by D.L.